Published by Institute of Physics Publishing for SISSA/ISAS

Received: *February 19, 2003*
Accepted: *March 2, 2003*

Exterior differentials in superspace and Poisson brackets

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Abstract: It is shown that two definitions for an exterior differential in superspace, giving the same exterior calculus, yet lead to different results when applied to the Poisson bracket. A prescription for the transition with the help of these exterior differentials from the given Poisson bracket of definite Grassmann parity to another bracket is introduced. It is also indicated that the resulting bracket leads to generalization of the Schouten-Nijenhuis bracket for the cases of superspace and brackets of diverse Grassmann parities. It is shown that in the case of the Grassmann-odd exterior differential the resulting bracket is the bracket given on exterior forms. The above-mentioned transition with the use of the odd exterior differential applied to the linear even/odd Poisson brackets, that correspond to semi-simple Lie groups, results, respectively, in also linear odd/even brackets which are naturally connected with the Lie superalgebra. The latter contains the BRST and anti-BRST charges and can be used for calculation of the BRST operator cogomology.

Keywords: BRST Quantization, BRST Symmetry, Superspaces, Differential and Algebraic Geometry.

Dedicated to the memory of Aleksandr Ivanovich Soroka and Raisa Artemovna Soroka (Gelukh)

http://jhep.sissa.it/archive/papers/jhep032003001/jhep032003001.pdf

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1. Introduction: two definitions of an exterior differential in superspace

There exist two possibilities to define an exterior differential in superspace with coordinates z^a having Grassmann parities $g(z^a) \equiv g_a$ and satisfying the permutation relations

$$z^a z^b = (-1)^{g_a g_b} z^b z^a \, .$$

The first one realized when we set the Grassmann parity of the exterior differential $d_0 = d_0 z^a \partial_{z^a}$ to be equal to zero

$$g(d_0 z^a) = g_a \, ,$$

where $\partial_{z^a} \equiv \partial / \partial z^a$. In this case the symmetry property of an exterior product of two differentials is

$$d_0 z^a \wedge d_0 z^b = (-1)^{g_a g_b + 1} d_0 z^b \wedge d_0 z^a \tag{1.1}$$

and a permutation relation of the exterior differential $d_0 z^a$ with the coordinate z^b has the form

$$d_0 z^a z^b = (-1)^{g_a g_b} z^b d_0 z^a \, .$$

Note that relation (1.1) can be rewritten in the following form

$$(-1)^{g_a} d_0 z^a \wedge d_0 z^b = (-1)^{(g_a+1)(g_b+1)} (-1)^{g_b} d_0 z^b \wedge d_0 z^a \, . \tag{1.2}$$

By defining an exterior product of a differential p-form

$$\Phi^0 = d_0 z^{a_p} \wedge \cdots \wedge d_0 z^{a_1} \phi^0_{a_1 \ldots a_p} \, , \qquad g(\phi^0_{a_1 \ldots a_p}) = g_{a_1} + \cdots + g_{a_p}$$

and a q-form

$$\Psi^0 = d_0 z^{b_q} \wedge \cdots \wedge d_0 z^{b_1} \psi^0_{b_1 \ldots b_q} \, , \qquad g(\psi^0_{b_1 \ldots b_q}) = g_{b_1} + \cdots + g_{b_q}$$

as

$$\Phi^0 \wedge \Psi^0 = (-1)^{pq} d_0 z^{b_q} \wedge \cdots \wedge d_0 z^{b_1} \wedge d_0 z^{a_p} \wedge \cdots \wedge d_0 z^{a_1} \phi^0_{a_1 \ldots a_p} \psi^0_{b_1 \ldots b_q} \, ,$$

we obtain the following symmetry property of this product

$$\Phi^0 \wedge \Psi^0 = (-1)^{pq} \Psi^0 \wedge \Phi^0 \, .$$

By setting the exterior differential of the p-form Φ^0 as follows

$$d_0\Phi^0 = d_0 \wedge d_0 z^{a_p} \wedge \cdots \wedge d_0 z^{a_1} \phi^0_{a_1\ldots a_p} = (-1)^p d_0 z^{a_p} \wedge \cdots \wedge d_0 z^{a_1} \wedge d_0 z^b \partial_{z^b} \phi^0_{a_1\ldots a_p}\,, \quad (1.3)$$

we obtain the Leibnitz rule for the exterior differential of the exterior product of two forms

$$d_0\left(\Phi^0 \wedge \Psi^0\right) = \left(d_0\Phi^0\right) \wedge \Psi^0 + (-1)^p \Phi^0 \wedge \left(d_0\Psi^0\right).$$

Note that very often another definition for the exterior differential is adopted

$$\overline{d_0\Phi^0} = d_0 z^{a_1} \wedge \cdots \wedge d_0 z^{a_p} \wedge d_0 z^b \partial_{z^b} \phi^0_{a_p\ldots a_1}$$

which differs from (1.3) with the absence of the grading factor $(-1)^p$ and leads to the following form of the Leibnitz rule

$$\overline{d_0(\Phi^0 \wedge \Psi^0)} = (-1)^q (\overline{d_0\Phi^0}) \wedge \Psi^0 + \Phi^0 \wedge (\overline{d_0\Psi^0})\,.$$

Another definition for the exterior differential in superspace arises when the Grassmann parity of the exterior differential $d_1 = d_1 z^a \partial_{z^a}$ is chosen to be equal to unit

$$g(d_1 z^a) = g_a + 1\,.$$

Then the symmetry property of the exterior product for two differentials is defined as[1]

$$d_1 z^a \tilde{\wedge} d_1 z^b = (-1)^{(g_a+1)(g_b+1)} d_1 z^b \tilde{\wedge} d_1 z^a \quad (1.4)$$

and a rule for the permutation of such a differential with the coordinate z^b has to be

$$d_1 z^a z^b = (-1)^{(g_a+1)g_b} z^b d_1 z^a\,.$$

Relation (1.4) can be represented in the form

$$(-1)^{g_a} d_1 z^a \tilde{\wedge} d_1 z^b = (-1)^{g_a g_b + 1} (-1)^{g_b} d_1 z^b \tilde{\wedge} d_1 z^a\,. \quad (1.5)$$

If the exterior product of the p-form

$$\Phi^1 = d_1 z^{a_p} \tilde{\wedge} \cdots \tilde{\wedge} d_1 z^{a_1} \phi^1_{a_1\ldots a_p}\,, \qquad g(\phi^1_{a_1\ldots a_p}) = g_{a_1} + \cdots + g_{a_p}$$

and q-form

$$\Psi^1 = d_1 z^{b_q} \tilde{\wedge} \cdots \tilde{\wedge} d_1 z^{b_1} \psi^1_{b_1\ldots b_q}\,, \qquad g(\psi^1_{b_1\ldots b_q}) = g_{b_1} + \cdots + g_{b_q}$$

is defined in the following way

$$\Phi^1 \tilde{\wedge} \Psi^1 = (-1)^{p(q+g_{b_1}+\cdots+g_{b_q})} d_1 z^{b_q} \tilde{\wedge} \cdots \tilde{\wedge} d_1 z^{b_1} \tilde{\wedge} d_1 z^{a_p} \tilde{\wedge} \cdots \tilde{\wedge} d_1 z^{a_1} \phi^1_{a_1\ldots a_p} \psi^1_{b_1\ldots b_q}\,,$$

then the symmetry property of this product is

$$\Phi^1 \tilde{\wedge} \Psi^1 = (-1)^{pq} \Psi^1 \tilde{\wedge} \Phi^1\,.$$

[1] In this case we use another notation $\tilde{\wedge}$ for the exterior multiplication.

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Let us define in this case the exterior differential of the p-form Φ^1 in the following form

$$d_1\Phi^1 = d_1\tilde{\wedge} d_1 z^{a_p}\tilde{\wedge}\cdots\tilde{\wedge} d_1 z^{a_1}\phi^1_{a_1\ldots a_p} = (-1)^{p+g_{a_1}+\cdots+g_{a_p}} d_1 z^{a_p}\tilde{\wedge}\cdots\tilde{\wedge} d_1 z^{a_1}\tilde{\wedge} d_1 z^b \partial_{z^b}\phi^1_{a_1\ldots a_p}\,.$$

Then the Leibniz rule for the exterior differential of the exterior product of a p-form Φ^1 and a q-form Ψ^1 will be

$$d_1\left(\Phi^1\tilde{\wedge}\Psi^1\right) = \left(d_1\Phi^1\right)\tilde{\wedge}\Psi^1 + (-1)^p\Phi^1\tilde{\wedge}\left(d_1\Psi^1\right).$$

Note that in this case the symmetry properties of the exterior product (1.4) coincide with the ones for the usual Grassmann product of two differentials for the coordinate z^a

$$d_1 z^a d_1 z^b = (-1)^{(g_a+1)(g_b+1)} d_1 z^b d_1 z^a\,. \tag{1.6}$$

The equivalence of the exterior calculi obtained with the use of the above mentioned different definitions for the exterior differential can be established as a result of the direct verification by taking into account relations (1.1) and (1.5) and by putting the following relations between coefficients of the corresponding p-forms Φ^0 and Φ^1

$$\phi^0_{a_1\ldots a_p} = (-1)^{\sum_{k=1}^{[p/2]} g_{a_{2k}}}\phi^1_{a_1\ldots a_p}\,,$$

where $[p/2]$ denotes a whole part of the quantity $p/2$. Thus, we proved that two definitions for the exterior differential, differed with the Grassmann parities, result in the same exterior calculus.

2. Poisson brackets related with the exterior differentials

Now we show that application of these differentials leads, however, to the different results under construction from a given Poisson bracket with a Grassmann parity $\epsilon = 0, 1$ of another one.

A Poisson bracket, having a Grassmann parity ϵ, written in arbitrary non-canonical variables z^a

$$\{A, B\}_\epsilon = A\,\overleftarrow{\partial}_{z^a}\,\omega^{ab}_\epsilon(z)\,\overrightarrow{\partial}_{z^b}\,B \tag{2.1}$$

has the following main properties:

$$\begin{aligned}
&g(\{A,B\}_\epsilon) \equiv g_A + g_B + \epsilon \pmod 2\,,\\
&\{A,B\}_\epsilon = -(-1)^{(g_A+\epsilon)(g_B+\epsilon)}\{B,A\}_\epsilon\,,\\
&\sum_{(ABC)}(-1)^{(g_A+\epsilon)(g_C+\epsilon)}\{A,\{B,C\}_\epsilon\}_\epsilon = 0\,,
\end{aligned}$$

which lead to the corresponding relations for the matrix ω^{ab}_ϵ

$$g\left(\omega^{ab}_\epsilon\right) \equiv g_a + g_b + \epsilon \pmod 2\,, \tag{2.2}$$

$$\omega^{ab}_\epsilon = -(-1)^{(g_a+\epsilon)(g_b+\epsilon)}\omega^{ba}_\epsilon\,, \tag{2.3}$$

$$\sum_{(abc)}(-1)^{(g_a+\epsilon)(g_c+\epsilon)}\omega^{ad}_\epsilon\partial_{z^d}\omega^{bc}_\epsilon = 0\,, \tag{2.4}$$

where $g_A \equiv g(A)$ and a sum with a symbol (abc) under it designates a summation over cyclic permutations of a, b and c.

The Hamilton equations for the phase variables z^a, which correspond to a hamiltonian H_ϵ $(g(H_\epsilon)=\epsilon)$,

$$\frac{dz^a}{dt}=\{z^a,H_\epsilon\}_\epsilon=\omega_\epsilon^{ab}\partial_{z^b}H_\epsilon \tag{2.5}$$

can be represented in the form

$$\frac{dz^a}{dt}=\omega_\epsilon^{ab}\partial_{z^b}H_\epsilon\equiv\omega_\epsilon^{ab}\frac{\partial(d_\zeta H_\epsilon)}{\partial(d_\zeta z^b)}\stackrel{\rm def}{=}(z^a,d_\zeta H_\epsilon)_{\epsilon+\zeta}\,,$$

where d_ζ $(\zeta=0,1)$ is one of the exterior differentials d_0 or d_1. By taking the exterior differential d_ζ from the Hamilton equations (2.5), we obtain

$$\frac{d(d_\zeta z^a)}{dt}=\left(d_\zeta\omega_\epsilon^{ab}\right)\frac{\partial(d_\zeta H_\epsilon)}{\partial(d_\zeta z^b)}+(-1)^{\zeta(g_a+\epsilon)}\omega_\epsilon^{ab}\partial_{z^b}(d_\zeta H_\epsilon)\stackrel{\rm def}{=}(d_\zeta z^a,d_\zeta H_\epsilon)_{\epsilon+\zeta}\,.$$

As a result of two last equations we have by definition the following binary composition for functions F and G of the variables z^a and their differentials $d_\zeta z^a\equiv y_\zeta^a$

$$(F,G)_{\epsilon+\zeta}=F\left[\overleftarrow{\partial}_{z^a}\,\omega_\epsilon^{ab}\,\overrightarrow{\partial}_{y_\zeta^b}+(-1)^{\zeta(g_a+\epsilon)}\,\overleftarrow{\partial}_{y_\zeta^a}\,\omega_\epsilon^{ab}\,\overrightarrow{\partial}_{z^b}+\overleftarrow{\partial}_{y_\zeta^a}\,y_\zeta^c\left(\partial_{z^c}\omega_\epsilon^{ab}\right)\overrightarrow{\partial}_{y_\zeta^b}\right]G\,. \tag{2.6}$$

In consequence of the grading properties (2.2) for the matrix ω_ϵ^{ab} this composition has the Grassmann parity $\epsilon+\zeta$

$$g[(F,G)_{\epsilon+\zeta}]\equiv g_F+g_G+\epsilon+\zeta\pmod 2\,.$$

By using the symmetry property (2.3) of ω_ϵ^{ab} , we can establish the symmetry of the composition (2.6)

$$(F,G)_{\epsilon+\zeta}=-(-1)^{(g_F+\epsilon+\zeta)(g_G+\epsilon+\zeta)}(G,F)_{\epsilon+\zeta}\,.$$

At last, taking into account relations (2.3) and (2.4) for the matrix ω_ϵ^{ab}, we come to the Jacobi identities for this composition

$$\sum_{(EFG)}(-1)^{(g_E+\epsilon+\zeta)(g_G+\epsilon+\zeta)}(E,(F,G)_{\epsilon+\zeta})_{\epsilon+\zeta}=0\,.$$

We see that the composition (2.6) satisfies all the main properties for the Poisson bracket with the Grassmann parity equal to $\epsilon+\zeta$. Thus, the application of the exterior differentials of opposite Grassamann parities to the given Poisson bracket results in the brackets of the different Grassmann parities.

Note that by transition to the variables $y_a^{\epsilon+\zeta}$, related with y_ζ^a by means of the matrix ω_ϵ^{ab}

$$y_\zeta^a=y_b^{\epsilon+\zeta}\omega_\epsilon^{ba}\,, \tag{2.7}$$

the Poisson bracket (2.6) takes a canonical form[2]

$$(F,G)_{\epsilon+\zeta}=F\left[\overleftarrow{\partial}_{z^a}\overrightarrow{\partial}_{y_a^{\epsilon+\zeta}}-(-1)^{g_a(g_a+\epsilon+\zeta)}\,\overleftarrow{\partial}_{y_a^{\epsilon+\zeta}}\overrightarrow{\partial}_{z^a}\right]G$$

that can be proved with the use of the Jacobi identities (2.4).

[2]There is no summation over ϵ in relation (2.7).

In the case $\zeta = 1$, due to relations (1.4), (1.6), the terms in the decomposition of a function $F(z^a, y^a_1)$ into degrees p of the variables y^a_1 can be treated as p-forms and the bracket (2.6) can be considered as a Poisson bracket on p-forms so that being taken between a p-form and a q-form results in a $(p+q-1)$-form[3]. The bracket (2.6) is a generalization of the bracket introduced in [2, 3] on the superspace case and on the case of the brackets (2.1) with arbitrary Grassmann parities.

Let us also note that if we take the bracket (2.6) in the component form and rise low indexes with the use of the matrix ω^{ab}_ϵ according to the rule (2.7) then we come to the generalizations of the Schouten-Nijenhuis brackets [4, 5] (see also [3, 6, 7, 8, 9, 10, 11]) onto the cases of superspace and the brackets of diverse Grassmann parities. The details of this generalization will be given in a separate paper [12].

It follows from the structure of the bracket (2.6) that if the initial bracket (2.1) is degenerate and possessed of a Casimir function $C(z)$

$$\{\ldots, C\}_\epsilon = 0\,,$$

then the bracket (2.6) has as Casimir functions this one

$$(\ldots, C)_{\epsilon+\zeta} = 0$$

and also a function of the form

$$\tilde{C} = y^a_\zeta \partial_{z^a} C, \quad (\ldots, \tilde{C})_{\epsilon+\zeta} = 0\,. \tag{2.8}$$

3. Linear Poisson brackets related with semi-simple Lie groups

Here we apply the procedure described in the previous section to the linear even and odd brackets connected with a semi-simple Lie group having structure constants $c_{\alpha\beta}{}^\gamma$ which obey the usual conditions

$$c_{\alpha\beta}{}^\gamma = -c_{\beta\alpha}{}^\gamma\,,$$
$$\sum_{(\alpha\beta\gamma)} c_{\alpha\beta}{}^\lambda c_{\lambda\gamma}{}^\delta = 0\,.$$

Let us take as an initial Poisson bracket (2.1) the linear even bracket given in terms of the commuting variables [4] x_α (here $z^a = x_\alpha$)

$$\{A, B\}_0 = A\, \overleftarrow{\partial}_{x_\alpha}\, c_{\alpha\beta}{}^\gamma x_\gamma\, \overrightarrow{\partial}_{x_\beta}\, B\,. \tag{3.1}$$

In the case of a semi-simple Lie group, which hereafter will be considered, this bracket has a Casimir function

$$C_0 = x_\alpha x_\beta g^{\alpha\beta}\,, \qquad \{\ldots, C_0\}_0 = 0\,, \tag{3.2}$$

where $g^{\alpha\beta}$ is an inverse tensor to the Cartan–Killing metric

$$g_{\alpha\beta} = c_{\alpha\gamma}{}^\lambda c_{\beta\lambda}{}^\gamma\,.$$

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[3] Concerning Poisson bracket between 1-forms and its relation with Lie bracket of vector fields see in the book [1].

[4] Lie-Poisson-Kirillov bracket.

By using the odd exterior differential d_1, we obtain from the bracket (3.1) in conformity with the transition from the bracket (2.1) to the bracket (2.6) the following linear odd bracket

$$(F,G)_1 = F\left(\overleftarrow{\partial}_{x_\alpha}\, c_{\alpha\beta}{}^{\gamma}x_\gamma\, \overrightarrow{\partial}_{\theta_\beta} + \overleftarrow{\partial}_{\theta_\alpha}\, c_{\alpha\beta}{}^{\gamma}x_\gamma\, \overrightarrow{\partial}_{x_\beta} + \overleftarrow{\partial}_{\theta_\alpha}\, c_{\alpha\beta}{}^{\gamma}\theta_\gamma\, \overrightarrow{\partial}_{\theta_\beta}\right) G\,, \tag{3.3}$$

where $\theta_\alpha = d_1 x_\alpha$ are Grassmann variables. In this case relation (2.7) has the form

$$\theta_\alpha = \Theta^\beta c_{\beta\alpha}{}^{\gamma} x_\gamma\,,$$

where Θ^β are also Grassmann variables in term of which the odd bracket (3.3) takes a canonical form

$$(F,G)_1 = F\left(\overleftarrow{\partial}_{x_\alpha}\overrightarrow{\partial}_{\Theta^\alpha} - \overleftarrow{\partial}_{\Theta^\alpha}\overrightarrow{\partial}_{x_\alpha}\right)G\,.$$

According to (2.8) the bracket (3.3) has as Casimir functions apart from C_0 (3.2) a nilpotent quantity

$$\tilde{C}_1 = x_\alpha\theta_\beta g^{\alpha\beta}\,, \qquad \left(\ldots,\tilde{C}_1\right)_1 = 0\,, \qquad (\tilde{C}_1)^2 = 0\,.$$

The odd bracket (3.3) has two nilpotent Batalin-Vilkovisky type differential second order Δ-operators [13, 14] (see also [15, 16, 17])

$$\Delta_{-1} = -\frac{1}{2}[\partial_{x_\alpha}(x_\alpha,)_1 + \partial_{\theta_\alpha}(\theta_\alpha,)_1] = -\frac{1}{2}S_\alpha\partial_{\theta_\alpha}\,, \qquad (\Delta_{-1})^2 = 0 \tag{3.4}$$

and

$$\Delta = -\frac{1}{2}[\partial_{x_\alpha}(x_\alpha,)_1 - \partial_{\theta_\alpha}(\theta_\alpha,)_1] = \left(T_\alpha + \frac{1}{2}S_\alpha\right)\partial_{\theta_\alpha}\,, \qquad \Delta^2 = 0\,, \tag{3.5}$$

where

$$T_\alpha = c_{\alpha\beta}{}^{\gamma}x_\gamma\partial_{x_\beta} \tag{3.6}$$

and

$$S_\alpha = c_{\alpha\beta}{}^{\gamma}\theta_\gamma\partial_{\theta_\beta} \tag{3.7}$$

are generators of the Lie group in the co-adjoint representation which obey the commutation relations

$$\begin{aligned} [T_\alpha, T_\beta] &= c_{\alpha\beta}{}^{\gamma}T_\gamma\,,\\ [S_\alpha, S_\beta] &= c_{\alpha\beta}{}^{\gamma}S_\gamma\,,\\ [T_\alpha, S_\beta] &= 0\,. \end{aligned}$$

Note that

$$(\theta_\alpha,)_1 = T_\alpha + S_\alpha \equiv Z_\alpha$$

and Z_α satisfy the relations

$$[Z_\alpha, Z_\beta] = c_{\alpha\beta}{}^{\gamma}Z_\gamma\,. \tag{3.8}$$

Now let us take as an initial bracket (2.1) the linear odd bracket introduced in refs. [18, 19] and given in terms of Grassmann variables θ_α (in this case $z^a = \theta_\alpha$)

$$\{A,B\}_1 = A\,\overleftarrow{\partial}_{\theta_\alpha}\, c_{\alpha\beta}{}^{\gamma}\theta_\gamma\, \overrightarrow{\partial}_{\theta_\beta}\, B\,, \tag{3.9}$$

which has in the case of the semi-simple Lie group a nilpotent Casimir function

$$C_1 = \theta^\alpha \theta^\beta \theta^\gamma c_{\alpha\beta\gamma}\,, \qquad \{\ldots, C_1\}_1 = 0\,, \qquad (C_1)^2 = 0\,, \tag{3.10}$$

where $\theta^\alpha = g^{\alpha\beta}\theta_\beta$ and $c_{\alpha\beta\gamma} = c_{\alpha\beta}{}^\lambda g_{\lambda\gamma}$. With the help of the odd differential d_1, according to the transition from (2.1) to (2.6), we come from the bracket (3.9) to the even linear bracket of the form

$$(F,G)_0 = F\left(\overleftarrow{\partial}_{\theta_\alpha}\, c_{\alpha\beta}{}^\gamma \theta_\gamma\, \overrightarrow{\partial}_{x_\beta} + \overleftarrow{\partial}_{x_\alpha}\, c_{\alpha\beta}{}^\gamma \theta_\gamma\, \overrightarrow{\partial}_{\theta_\beta} + \overleftarrow{\partial}_{x_\alpha}\, c_{\alpha\beta}{}^\gamma x_\gamma\, \overrightarrow{\partial}_{x_\beta}\right) G\,, \tag{3.11}$$

where $x_\alpha = d_1\theta_\alpha$ are commuting variables. Relation (2.7) in this case has the form

$$x_\alpha = \Theta^\beta c_{\beta\alpha}{}^\gamma \theta_\gamma\,,$$

where Θ^β are Grassmann variables in term of which the bracket (3.11) takes a canonical form for the Martin bracket [20]

$$(F,G)_0 = F(\overleftarrow{\partial}_{\theta_\alpha}\overrightarrow{\partial}_{\Theta^\alpha} + \overleftarrow{\partial}_{\Theta^\alpha}\overrightarrow{\partial}_{\theta_\alpha})G\,.$$

In accordance with (2.8) the even bracket (3.11) has as Casimir functions besides C_1 (3.10) the function

$$\tilde{C}_0 = x^\alpha \theta^\beta \theta^\gamma c_{\alpha\beta\gamma}, \quad (\ldots, \tilde{C}_0)_0 = 0\,,$$

where $x^\alpha = g^{\alpha\beta}x_\beta$.

The even bracket (3.11), in contrast to the odd bracket, has no second order nilpotent differential Δ-like operators. It is surprising enough that instead of this it has two *nilpotent* differential operators of the first order

$$\Delta_1 = -\frac{1}{2}[\theta^\alpha(x_\alpha,)_0 + x^\alpha(\theta_\alpha,)_0] = -\frac{1}{2}\theta^\alpha S_\alpha\,, \qquad (\Delta_1)^2 = 0\,,$$

and

$$Q = \frac{1}{2}[\theta^\alpha(x_\alpha,)_0 - x^\alpha(\theta_\alpha,)_0] = \theta^\alpha(T_\alpha + \frac{1}{2}S_\alpha)\,, \qquad Q^2 = 0\,. \tag{3.12}$$

In the papers [18, 19] the operators Δ_1 and Δ_{-1}, defined on the Grassmann algebra with generators θ_α, have been introduced in connection with the linear odd Poisson bracket (3.9), which corresponds to a semi-simple Lie group, and the Lie superalgebra for them has been given. These operators are standard terms in the BRST and anti-BRST charges respectively.

4. Lie superalgebra for the BRST and anti-BRST charges

Thus, in the superspace with coordinates x_α, θ_α with the help of the linear even (3.11) and odd (3.3) Poisson brackets we constructed the operators Q (3.12) and Δ (3.5). These operators can be treated as the BRST and anti-BRST charges correspondingly (see, e.g., [21]) if we consider θ^α and ∂_{θ^α} as representations for the ghosts and antighosts operators respectively. The operators Q and Δ satisfy the following anticommutation relation:

$$\{Q, \Delta\} = \frac{1}{2}(T^\alpha T_\alpha + Z^\alpha Z_\alpha)\,, \tag{4.1}$$

two terms in the right-hand side of which, because of the commutation relations

$$[T^\alpha T_\alpha, Q] = 0\,, \tag{4.2}$$
$$[T^\alpha T_\alpha, \Delta] = 0\,, \tag{4.3}$$
$$[Z_\alpha, Q] = 0\,, \tag{4.4}$$
$$[Z_\alpha, \Delta] = 0\,, \tag{4.5}$$
$$[T^\alpha T_\alpha, Z_\beta] = 0\,, \tag{4.6}$$

are central elements of the Lie superalgebra formed by the quantities Q, Δ, $T^\alpha T_\alpha$ and $Z^\alpha Z_\alpha$. The relations (4.1)–(4.6) remain valid if we take instead of the co-adjoint representation (3.6) an arbitrary representation for the generators T_α. The quantity $Z^\alpha Z_\alpha$ contains the term $S^\alpha S_\alpha$ which can be written as

$$S^\alpha S_\alpha = N - K$$

where

$$N = \theta^\alpha \partial_{\theta^\alpha}\,, \tag{4.7}$$

can be considered as a representation for the ghost number operator and the quantity K has the form

$$K = \frac{1}{2}\theta^\alpha\theta^\beta c_{\alpha\beta}{}^\lambda c_{\lambda\gamma\delta}\partial_{\theta_\gamma}\partial_{\theta_\delta}\,.$$

The operator N has the following permutation relations with Q and Δ:

$$[N, Q] = Q\,, \tag{4.8}$$
$$[N, \Delta] = -\Delta \tag{4.9}$$

and commutes with the central elements $T^\alpha T_\alpha$ and $Z^\alpha Z_\alpha$

$$[N, T_\alpha] = 0\,, \tag{4.10}$$
$$[N, Z_\alpha] = 0\,. \tag{4.11}$$

We can add the commutation relations (3.8) for the generators Z_α with the usual quadratic Casimir operator $Z^\alpha Z_\alpha$ for the semi-simple Lie group.

Note that the Lie superalgebra for the quantities Q, Δ, N, $T^\alpha T_\alpha$ and $Z^\alpha Z_\alpha$ determined by the relations (3.8), (4.1)–(4.11) can be used for the calculation of the BRST operator cohomologies [22].

5. Conclusion

Thus, we illustrated that in superspace the exterior differentials with opposite Grassmann parities give the same exterior calculus.

Then we introduced a prescription for the construction with the help of these differentials from a given Poisson bracket of the definite Grassmann parity of another one. We showed that the parity of the resulting bracket depends on the parities of the both initial bracket and exterior differential used.

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It is also indicated that the resulting bracket is related with a generalization of the Schouten-Nijenhuis bracket on the superspace case and on the brackets of an arbitrary Grassmann parity.

By applying the prescription to the linear odd and even Poisson brackets, corresponding to a semi-simple Lie group with the structure constants $c_{\alpha\beta}{}^{\gamma}$ and given respectively on the anticommuting θ_α and commuting x_α variables, we come with the help of the Grassmann-odd exterior differential to the correspondingly even and odd linear Poisson brackets which are both defined on the superspace with the coordinates x_α, θ_α and related also with the same semi-simple Lie group.

We demonstrated that these resulting even and odd brackets are naturally connected with the BRST and anti-BRST charges respectively.

Acknowledgments

One of the authors (V.A.S.) is grateful to A.P. Isaev, J. Lukierski, M. Tonin and J. Wess for useful discussions and to S.J. Gates, Jr., P. Van Nieuwenhuizen, W. Siegel and B. Zumino for stimulating discussions and for hospitality respectively at the University of Maryland, SUNY (Stony Brook) and LBNL (Berkeley) where the parts of the work have been performed. V.A.S. thanks L. Bonora for fruitful discussions and for hospitality at SISSA (Trieste) where this work has been completed.

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JHEP03(2003)001